Applications of Uniform Sampling: Densest Subgraph and Beyond*

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Abstract

Recently [Bhattacharya et al., STOC 2015] [3] provide the first non-trivial algorithm for the densest subgraph problem in the streaming model with additions and deletions to its edges, i.e., for dynamic graph streams. They present a $(0.5-\epsilon)$ -approximation algorithm using $\tilde{O}(n)$ space, where factors of ϵ and $\log(n)$ are suppressed in the \tilde{O} notation. However, the update time of this algorithm is large. To remedy this, they also provide a $(0.25-\epsilon)$ -approximation algorithm using $\tilde{O}(n)$ space with update time $\tilde{O}(1)$.

In this paper we improve the algorithms in [3] by providing a $(1-\epsilon)$ -approximation algorithm using $\tilde{O}(n)$ space. Our algorithm is conceptually simple - it samples $\tilde{O}(n)$ edges uniformly at random, and finds the densest subgraph on the sampled graph. We also show how to perform this sampling with update time $\tilde{O}(1)$. In addition to this, we show that given oracle access to the edge set, we can implement our algorithm in time $\tilde{O}(n)$ on a graph in the standard RAM model. To the best of our knowledge this is the fastest $(0.5 - \epsilon)$ -approximation algorithm for the densest subgraph problem in the RAM model given such oracle access.

Interestingly, we extend our results to a general class of graph optimization problems that we call heavy subgraph problems. This class contains many interesting problems such as densest subgraph, directed densest subgraph, densest bipartite subgraph, d-max cut, and d-sum-max clustering. Our results, by characterizing heavy subgraph problems, address Open Problem 13 at the IITK Workshop on Algorithms for Data Streams in 2006 regarding the effects of subsampling, in the context of graph streams.

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1 Introduction

In this paper we consider a general class of graph optimization problems that we call heavy subgraph problems in the streaming setting with additions and deletions, i.e., in dynamic graph streams. We show that many interesting problems such as densest subgraph, directed densest subgraph, densest bipartite subgraph, d-max cut, and d-sum-max clustering fit in this general class of problems. To the best of our knowledge, we are the first to consider densest bipartite subgraph and d-sum-max clustering in the streaming setting. We defer the definitions of these two problems to Subsections 4.1 and 4.4, respectively.

Finding the densest subgraph of a graph is one of the classical problems in computer science and appears to have many applications that deal with massive data such as spam detection [9] and analyzing communication in social networks [6].

In an instance of the densest subgraph problem we are given a graph G and want to find a subgraph $sol \subseteq G$, that maximizes $\frac{|E_{sol}|}{|V_{sol}|}$, where V_{sol} and E_{sol} are the vertex set and the edge set of sol, respectively. Similarly, in an instance of the directed densest subgraph problem we are given a directed graph $G = (V_G, E_G)$ and want to find a pair $A, B \subseteq V_G$, that maximizes $\frac{|E(A,B)|}{\sqrt{|A| \cdot |B|}}$, where E(A,B) is the number of edges from A to B in E_G .

Charikar [4] studies the densest subgraph problem in the classical setting and provides a 0.5-approximation algorithm with running time O(n+m), where n and m are the number of vertices and edges, respectively. To the best of our knowledge, this is the fastest known constant approximation algorithm for the densest subgraph problem. Moreover, he provides a 0.5-approximation algorithm for the directed densest subgraph problem.

Later, Bahmani, Kumar and Vassilvitskii [2], consider the densest subgraph problem and the directed densest subgraph problem in the streaming setting with only insertions of edges. For both problems, they present streaming algorithms with a $\frac{0.5}{1+\epsilon}$ -approximation factor using $\log_{1+\epsilon}(n)$ passes over the input. To the best of our knowledge, their results for directed graphs are the only non-trivial results for the directed densest subgraph problem in the streaming setting, prior to our work.

Recently, Bhattacharya et al. [3] present the first single pass streaming algorithm for dynamic graph streams. Their first algorithm provides a $(0.5 - \epsilon)$ -approximation using $\tilde{O}(n)$ bits of space, though the update time is inefficient. They provide a second algorithm with a $(0.25 - \epsilon)$ -approximation factor for which the update time is only $\tilde{O}(1)$, again using $\tilde{O}(n)$ bits of space.

In the d-max cut problem, we are given a graph G, and we want to decompose the vertices of G into d partitions such that the number of edges between different partitions is maximized. To the best of our knowledge, we are the first to consider this problem for general d in the streaming setting. A restricted version of this problem where d=2, is the classic max cut problem. One can store a sparsifier [1] of the input graph in $\tilde{O}(n)$ space, and preserve a $(1-\epsilon)$ -approximation of the max cut. However, it is not clear if sparsifiers preserve d-max cut or not. Recently, Kapralov, Khanna and Sudan [13] show that any $(1-\epsilon)$ -approximation to the max cut problem in the streaming setting requires $n^{1-O(\epsilon)}$ space.

In this paper, we refer to $\frac{|E_{sol}|}{|V_{sol}|}$ as the density of the solution sol, and denote it by den(sol). We refer to the density of the densest subgraph of G by opt(G). We also denote the densest subgraph among all those on k vertices by opt(G,k). We sometimes abuse notation and use opt(G) to refer to both the densest subgraph as well as its density. It is easy to see that the densest subgraph of a graph G is an induced subgraph of it, otherwise we can simply add more edges to it. Therefore,

we can indicate the densest subgraph by its vertex set. For a graph G, and a subset of vertices U, we denote the induced subgraph of G on vertex set U by G[U].

1.1 Our Results

In this paper, we first consider the densest subgraph problem in the streaming setting where we have both insertions and deletions to the edges as they arrive in the stream, i.e., in a dynamic graph stream. We improve the result of Bhattacharya et al. (STOC'15) [3] by providing a $(1 - \epsilon)$ -approximation algorithm for this problem using $\tilde{O}(n)$ space. Indeed, our algorithm simply samples $\tilde{O}(n)$ edges uniformly at random, and finds the densest subgraph on the sampled graph. We also achieve update time $\tilde{O}(1)$. To achieve this, we use min-wise independent hashing together with fast multi-point polynomial evaluation.

Theorem 1.1. There exists a semi-streaming algorithm in dynamic graph streams for the densest subgraph problem with space $\tilde{O}(n)$ which gives a $(1-\epsilon)$ -approximate solution, with probability 1-1/n. The update time is $\tilde{O}(1)$.

In addition, our algorithm can be implemented using an oracle that provides direct access to a uniformly sampled edge. This algorithm makes $\tilde{O}(n)$ queries to the oracle and finds the densest subgraph on a graph with n vertices and $\tilde{O}(n)$ edges. Therefore, by replacing the m in the running time of the algorithm of Charikar [4] with $\tilde{O}(n)$, we achieve $\tilde{O}(n)$ running time, which is tight up to logarithmic factors. To the best of our knowledge, this is the fastest constant approximation algorithm for the densest subgraph problem in the RAM model, given such an oracle.

Theorem 1.2. Suppose we have oracle access to the edge set of the input graph with the ability to sample an edge uniformly at random. There exists an algorithm for the densest subgraph problem running in time $\tilde{O}(n)$, which gives a $(0.5 - \epsilon)$ -approximate solution, with probability 1 - 1/n.

Next, we extend our results to a general family of graph optimization problems that we call heavy subgraph problems. Interestingly, we show that by uniformly sampling edges we obtain enough information about the solution of any heavy subgraph problem. Since the solution of a heavy subgraph problem itself may be as large as the whole graph, here we just claim that we can estimate the size of the optimum solution. However, in some cases, like for the densest subgraph problem, it might be possible to also obtain the optimum solution itself, and not just the size, from the sampled graph.

A graph optimization problem is defined by, an input graph G, a set of feasible solutions Sol_G , which are subgraphs of G, and an objective function $f: Sol \to \mathbb{R}$. In a graph optimization problem we aim to find a solution $sol \in Sol_G$ that maximizes f. In fact, the number of feasible solutions for a graph G may be exponential in the size of G. We say a graph optimization problem on graph G is a (γ, l) -heavy subgraph problem if there exist l sets $Sol_G^1, Sol_G^2, \ldots, Sol_G^l$, such that $Sol_G = \bigcup_{k=1}^l Sol_G^k$ and for any k:

- Local Linearity: There are l numbers $f_1 \geq f_2 \geq \cdots \geq f_l = 1$, such that for any $1 \leq k \leq l$ and any solution $sol \in Sol_G^k$, we have $f(sol) = f_k \cdot |E_{sol}|$, where E_{sol} is the edge set of sol.
- Hereditary Property: For any spanning subgraph $H \subseteq G$, we have $sol_H \in Sol_H^k$ if and only if there exists a solution $sol_G \in Sol_G^k$ such that $sol_H = sol_G \cap H$.
- γ Bound: γ is chosen such that the optimum solution is lower bounded by $\gamma \log(|Sol_G^k|) f_k \frac{m}{n}$.

Let $\mathcal{P}(\gamma, l)$ be a heavy subgraph problem, and let Alg be an α -approximation algorithm for \mathcal{P} . Algorithm 2 samples $O(\frac{n\delta \log(l)}{\gamma\epsilon^2})$ edges of the input graph and runs Alg on the sampled graph. Interestingly, the following Theorem shows Algorithm 2 is an $(\alpha - \epsilon)$ -approximation algorithm for \mathcal{P} on G.

Theorem 1.3. Let $\mathcal{P}(\gamma, l)$ be a heavy subgraph problem. Let G be an arbitrary graph G, and let Alg be an α -approximation algorithm for \mathcal{P} . With probability $1 - e^{-\delta}$, Algorithm 2 is an $(\alpha - \epsilon)$ -approximation algorithm for \mathcal{P} on G, using $O(\frac{n\delta \log(l)}{\gamma \epsilon^2})$ space.

Finally, in section 4, we show several applications of Theorem 1.3. Indeed, we show that directed densest subgraph, densest bipartite subgraph, d-max cut and d-sum-max clustering all fits in the general family of heavy subgraph problems, and thus, Theorem 1.3 holds for them.

Theorem 1.4. The following statements hold.

- Densest bipartite subgraph is a $(\gamma = \frac{2}{\log(n)+1}, l = n)$ heavy subgraph problem.
- Directed densest subgraph is a $(\gamma = \frac{1}{2\sqrt{n}\log(n)}, l = n^2)$ heavy subgraph problem.
- d-max cut is a $(\gamma = \frac{1}{2 \log(d)}, l = 1)$ heavy subgraph problem.
- d-sum-max clustering is a $(\gamma = \frac{n-2d}{n \log(d)}, l = 1)$ heavy subgraph problem.

In fact, understanding the structure of the problems that can be solved using sampling, and specifically uniform sampling, is a well-motivated challenge, which was highlighted as a direction in the IITK Workshop on Algorithms for Data Streams in 2006. Our structural results, as well as our characterization for heavy subgraph problems, give partial answers to this open question in the context of graphs.

In simultaneous and independent work McGregor et al. [14] present a single pass $(1 - \epsilon)$ -approximation algorithm for the densest sugraph problem in the dynamic graph streaming model with update time $\tilde{O}(1)$, that uses $\tilde{O}(n)$ space.

2 Densest Subgraph

In this section we analyze a simple algorithm for the densest subgraph problem. This algorithm simply samples $\tilde{O}(n)$ edges uniformly at random (without replacement) and solves the problem on the sampled subgraph, where factors of $1/\epsilon$, δ and $\log(n)$ are hidden in the \tilde{O} notation. Interestingly, we show that, with probability $1-m^{-\delta}$, this algorithm gives a $(1-\epsilon)$ -approximate solution using $\tilde{O}(n)$ bits of space.

In addition, this algorithm can be implemented with a running time of $\tilde{O}(n)$, using oracle access to the edge set, with the ability to sample an edge uniformly at random. This works by combining our sampling guarantee with the algorithm of Charikar [4].

To the best of our knowledge, our $(0.5 - \epsilon)$ -approximation algorithm is the fastest constant approximation algorithm for the densest subgraph problem. This algorithm can be implemented in the streaming setting with insertions and deletions to the edges (the *strict turnstile* or *dynamic graph* stream model), as well as the RAM model with oracle access to random edge samples (we discuss the latter model below in the context of improving the running time of the best offline algorithms).

Before stating our main lemma first we need the following generalized version of the Chernoff inequality, that holds for negatively correlated random variables. We say Boolean random variables x_1, x_2, \ldots, x_r are negatively correlated if for any arbitrary subset S of $\{x_1, x_2, \ldots, x_r\}$, and any arbitrary $a \in S$ we have $\Pr(a = 1 | \forall_{b \in S-a} b = 1) \leq \Pr(a = 1)$ [7].

Lemma 2.1 ([15]). Let x_1, x_2, \ldots, x_r be a sequence of negatively correlated boolean (i.e., 0 or 1) random variables, and let $X = \sum_{i=1}^{r} x_i$. We have

$$\Pr(|X - \mathbf{E}[X]| \ge \epsilon \mathbf{E}[X]) \le 3 \exp(-\epsilon^2 \mathbf{E}[X]/3).$$

To use the above generalized version of the Chernoff bound, we need to show that our random variables are negatively correlated. The following lemma becomes very useful to show the random variables are negatively correlated.

Lemma 2.2. Let $x_1, x_2, ..., x_r$ be a sequence of Boolean random variables, such that, exactly t of them are chosen to be 1 uniformly at random. Then the random variables $x_1, x_2, ..., x_r$ are negatively correlated.

Proof. Let S be an arbitrary subset of $\{x_1, x_2, \ldots, x_r\}$ and let a be an arbitrary element of S. On the one hand, the probability that a=1 is $\frac{t}{r}$. On the other hand, conditioned that for any element $b \in S \setminus \{a\}$, we have b=1; the probability that a=1 is $\frac{t-(|S|-1)}{r-(|S|-1)}$. Clearly, $\frac{t}{r} \geq \frac{t-(|S|-1)}{r-(|S|-1)}$, which gives us $\Pr(a=1|\forall_{b\in S-a}b=1) \leq \Pr(a=1)$. This means the random variables x_1, x_2, \ldots, x_r are negatively correlated.

Algorithm 1 Finding Densest Subgraph

Input: A graph $G = (E_G, V)$.

Output: A $(1-\epsilon)$ -approximation of the densest subgraph of G, w.pr. $1-m^{-\delta}$.

- 1: Set $C = \frac{12n(4+\delta)\log(m)}{\epsilon^2}$
- 2: if $|E| \leq C$ then
- 3: Find a densest subgraph of G using the algorithm of [4] (or any other algorithm).
- 4: **else**
- 5: Sample C edges uniformly at random, without replacement from G.
- 6: Let H be the sampled graph.
- 7: Find a densest subgraph of H.

Lemma 2.3. Let $G = (V, E_G)$ be a graph with vertex set V and edge set E_G . Let $H = (V, E_H)$ be the sampled graph in Algorithm 1 and let $p = \frac{C}{|E_G|}$. We have for any k,

$$\Pr\left(opt(G,k) - den_G(opt(H,k)) \ge \epsilon opt(G)\right) \le 6 \exp(k \cdot \log(m) - \frac{pk\epsilon^2 opt(G)}{12}),$$

where $den_G(opt(H, k))$ is the density of the subgraph of G induced by the vertices of opt(H, k).

Proof. Let x_e be the random variable that indicates whether e exists in E_H or not, and let U be an arbitrary subset of V of size k. By definition, the number of edges in H[U] is $\sum_{e \in H[U]} 1 =$

 $\sum_{e \in G[U]} x_e$. Let us denote this summation by X_U . Then, we have

$$E[den(H[U])] = E\left[\frac{\sum_{e \in G[U]} x_e}{|U|}\right] = \frac{E[\sum_{e \in G[U]} x_e]}{|U|} = \frac{\sum_{e \in G[U]} p}{|U|} = p \frac{\sum_{e \in G[U]} 1}{|U|} = p \cdot den(G[U])$$
(1)

where the first and the last equalities are by definition of density, and the third equality is by definition of p. Lemma 2.2 says that random variables $x_1, x_2, \ldots, x_{|E_G|}$ are negatively correlated. Thus, to bound X_U , we can apply the following form of the Chernoff bound from Lemma 2.1.

$$\Pr(|X - E[X]| \ge \epsilon' E[X]) \le 3 \exp(-\epsilon'^2 E[X]/3)$$

By setting $\epsilon' = pk\epsilon \frac{opt(G)}{2 \to [X]}$ we have

$$\Pr\left(|X - \mathbf{E}[X]| \ge \frac{pk\epsilon opt(G)}{2}\right) \le 3\exp\left(-\frac{p^2k^2\epsilon^2 opt(G)^2}{4\mathbf{E}[X]^2} \cdot \frac{\mathbf{E}[X]}{3}\right)$$

$$\le 3\exp\left(-\frac{p^2k^2\epsilon^2 opt(G)^2}{12\mathbf{E}[X]}\right)$$

$$\le 3\exp\left(-\frac{pk\epsilon^2 opt(G)}{12}\right)$$
Using Equality 1

On the other hand we have

$$\Pr\left(|X - \mathbf{E}[X]| \ge \frac{pk\epsilon opt(G)}{2}\right) = \Pr\left(\frac{1}{p} \frac{|X - \mathbf{E}[X]|}{k} \ge \frac{\epsilon}{2} opt(G)\right)$$

$$= \Pr\left(\left|\frac{1}{p} \frac{X}{k} - \frac{\frac{1}{p} \mathbf{E}[X]}{k}\right| \ge \frac{\epsilon}{2} opt(G)\right)$$

$$= \Pr\left(\left|\frac{1}{p} den(H[U]) - den(G[U])\right| \ge \frac{\epsilon}{2} opt(G)\right) \quad \text{Using Equality 1}$$

Therefore, we have

$$\Pr\left(\left|\frac{1}{p}den(H[U]) - den(G[U])\right| \ge \frac{\epsilon}{2}opt(G)\right) \le 3\exp(-\frac{pk\epsilon^2opt(G)}{12}). \tag{2}$$

If we set U to be the vertex set of Opt(G,k), Inequality 2 says that

$$\Pr\left(opt(G,k) - \frac{1}{p}den_H(opt(G,k)) \ge \frac{\epsilon}{2}opt(G)\right) \le 3\exp(-\frac{pk\epsilon^2opt(G)}{12})$$

which immediately gives us

$$\Pr\left(opt(G,k) - \frac{1}{p}opt(H,k) \ge \frac{\epsilon}{2}opt(G)\right) \le 3\exp(-\frac{pk\epsilon^2opt(G)}{12}). \tag{3}$$

On the other hand, Inequality 2 says that for each selection of U, with probability $1 - 3\exp(-\frac{pk\epsilon^2 opt(G)}{12})$, we can upper bound $\frac{1}{p}den(H[U])$ by $den(G[U]) + \frac{\epsilon}{2}opt(G)$. As we have $\binom{m}{k}$ such choices, applying a union bound we have

$$\Pr\left(\forall_{U:|U|=k} \frac{1}{p} den(H[U]) - den(G[U]) \ge \frac{\epsilon}{2} opt(G)\right) \le 3\binom{m}{k} \exp(-\frac{pk\epsilon^2 opt(G)}{12}).$$

If we set U to opt(H, k) we have

$$\Pr\left(\frac{1}{p}opt(H,k) - den_G(opt(H,k)) \ge \frac{\epsilon}{2}opt(G)\right) \le 3\binom{m}{k} \exp(-\frac{pk\epsilon^2 opt(G)}{12}). \tag{4}$$

Therefore, by combining Inequalities 3 and 4 and applying the union bound we have

$$\Pr\left(opt(G,k) - den_G(opt(H,k)) \ge \epsilon opt(G)\right) \le 3\left(\binom{m}{k} + 1\right) \exp\left(-\frac{pk\epsilon^2 opt(G)}{12}\right)$$
$$\le 3(m^k + 1) \exp\left(-\frac{pk\epsilon^2 opt(G)}{12}\right)$$
$$\le 6 \exp\left(k \cdot \log(m) - \frac{pk\epsilon^2 opt(G)}{12}\right).$$

In fact, the density of the densest subgraph of a graph G is at least as much as the density of G itself. Hence, one can lower bound opt(G) by the density of G which is $\frac{m}{n}$. The following states this fact.

Fact 2.4. The density of the densest subgraph of G is at least $\frac{m}{n}$, where n is the number of vertices in G and m is the number of edges.

The following theorem bounds the approximation ratio of Algorithm 1.

Theorem 2.5. With probability $1 - m^{-\delta}$, Algorithm 1 is a $(1 - \epsilon)$ -approximation algorithm for the densest subgraph problem.

Proof. Recall that Lemma 2.3 states for an arbitrary fixed k, with probability at least $1-6\exp(k\cdot\log(m)-\frac{pk\epsilon^2opt(G)}{12})$ we have $opt(G,k)-den_G(opt(H,k))\leq \epsilon opt(G)$. Using a union bound this holds for all $1\leq k\leq n$, with probability $1-\sum_{k=1}^n 6\exp(k\cdot\log(m)-\frac{pk\epsilon^2opt(G)}{12})$. Thus, for some value of k, with probability $1-n\cdot 6\exp(k\cdot\log(m)-\frac{pk\epsilon^2opt(G)}{12})$ we have $opt(G)-den_G(opt(H))\leq \epsilon opt(G)$, which means that Algorithm 1 returns a $(1-\epsilon)$ -approximation. Now, if we set p to $\frac{12n(4+\delta)\log(m)}{\epsilon^2m}$,

or equivalently set C in Algorithm 1 to $\frac{12n(4+\delta)\log(m)}{\epsilon^2}$, we have

$$1 - n \times 6 \exp(k \cdot \log(m) - \frac{pk\epsilon^2 opt(G)}{12})$$

$$= 1 - n \times 6 \exp(k \cdot \log(m) - \frac{12n(4+\delta)\log(m)}{\epsilon^2 m} \times \frac{k\epsilon^2 opt(G)}{12})$$

$$\geq 1 - n \times 6 \exp(k \cdot \log(m) - \frac{12n(4+\delta)\log(m)}{\epsilon^2 m} \times \frac{k\epsilon^2 m}{12n})$$
From Fact 2.4
$$= 1 - n \times 6 \exp(k \cdot \log(m) - (4+\delta)k \cdot \log(m))$$

$$= 1 - n \times 6 \exp(-(3+\delta)k \cdot \log(m))$$

$$\geq 1 - \exp(\log(n) + 2 - (3+\delta)k \cdot \log(m))$$

$$\geq 1 - \exp(-\delta \cdot k \cdot \log(m))$$

$$\geq 1 - \exp(-\delta \cdot k \cdot \log(m))$$

$$\geq 1 - \exp(-\delta \cdot \log(m))$$

$$= 1 - m^{-\delta}.$$

One can use L_0 -samplers [11] to sample the C edges in Algorithm 1 in a dynamic graph stream. However, maintaining these L_0 samplers may need an update time as large as C. In Lemma 2.9 we show how to sample C edges with $\tilde{O}(1)$ update time using the notion of min-wise independent hash functions.

Definition 2.6. Given $\epsilon > 0$, we say a hash function $h : [1, n] \to [1, n]$ is ϵ -approximately t-min-wise independent on a subset X of $\{1, 2, \ldots, n\}$ if for any $Y \subseteq X$ such that |Y| = t we have

$$Pr(\max_{y \in Y} h(y) < \min_{z \in X - Y} h(z)) = \frac{1}{\binom{|X|}{|Y|}} (1 \pm \epsilon).$$

Theorem 2.7. (Theorem 1.1 of [8]) There exist constants c', c'' > 1 such that for any $X \subseteq \{1, 2, ..., n\}$ of size at most $\epsilon n/c'$, any $c''(t \log \log(1/\epsilon) + \log(1/\epsilon))$ -wise independent family of functions h is ϵ -approximately t-min-wise independent on X.

In addition, one can evaluate h on t items simultaneously in total time $t \cdot (\log(t/\epsilon))^{O(1)}$ by using fast multipoint polynomial evaluation (see, e.g., Theorem 13 of [12], where this idea was used for a different streaming problem). Further, one can spread the evaluation of h on t items evenly across the next t stream updates, converting this amortized $(\log(t/\epsilon))^{O(1)}$ update time to a worst-case update time.

We need the following generalized version of the Chernoff bound.

Lemma 2.8 ([16]). Let X be the sum of t-wise independent Boolean random variables. For any $\epsilon \geq 1$ such that $t \geq |\epsilon^2 E[X]e^{-1/3}|$, we have

$$Pr(|X - E[X]| > \epsilon E[X]) < \exp(-\lfloor \epsilon^2 E[X]/3\rfloor).$$

Lemma 2.9. For any number $C \ge n$ of edge samples and constant $1 \le \delta$, we can sample C edges in a dynamic stream (a stream with insertions and deletions to the edges), such that the statistical distance of our sampled edges to a uniformly random (without replacement) sample of C edges is $O(e^{-\delta})$. This sampling algorithm uses $\tilde{O}(C)$ space and has update time $\tilde{O}(1)$.

Proof. We apply Theorem 2.7 with the n of that theorem equal to $\binom{n}{2}c'/\epsilon$, where ϵ is set to $e^{-\delta}$. We label each possible edge of our graph with a number in $\{1,2,\ldots,\binom{n}{2}\}$ and extend the domain and range of h to $\{1,2,\ldots,n^2c'/\epsilon\}$. Then, we apply the X of Theorem 2.7 to the specific subset of $m \leq \binom{n}{2}$ edges in our input graph. We apply Theorem 2.7 with $t = \Theta(C\delta)$. Theorem 2.7 and the definition of min-wise independence imply that the statistical distance of the subset Y of t minimum hash values of our edges under h is within $e^{-\delta}$ from t uniformly random samples without replacement. Indeed, the probability of choosing any set Y is $\frac{1}{\binom{|X|}{|Y|}}(1 \pm \epsilon)$ rather than $\frac{1}{\binom{|X|}{|Y|}}$ had we had full independence, so summing the absolute values of these differences gives ℓ_1 -distance at most $\epsilon = e^{-\delta}$, and so statistical distance at most $e^{-\delta}/2$ between the distributions. Note that Theorem 2.7 implies that h is also $\Theta(C\delta)$ -wise independent (in the standard sense, not the min-wise sense), and so h is $\Omega(n)$ -wise independent using our assumptions that $C \geq n$ and $\delta \geq 1$.

Now we show how to maintain the C edges with the smallest hash values under h in a dynamic stream. We use $\log(n^2)$ sparse recovery data structures, $sp_1, sp_2, \dots sp_{2\log(n)}$ each to recover $9\delta C$ edges. Note that n^2 is an upper bound on the total number of distinct edges in our graph. Upon the update (insertion or deletion) of an edge e, we update each sparse recovery structure s_i such that $i \in [1, \lfloor \log_2(h(e)) \rfloor]$. For the sparse recovery structure, we use the data structure of [10], which has $\tilde{O}(1)$ update time, space $\tilde{O}(C)$ and succeeds with probability $1 - 1/n^2$ in returning all of the non-zero items in a vector for which it is applied to, provided this number of non-zero items is at most $9\delta C$.

Let Z_i be the number of distinct edges which hash under h to the i-th sparse recovery data structure sp_i . Then if $m \geq 4\delta C$, there always exists one of the sp_i such that $4\delta C \leq \mathrm{E}[Z_i] \leq 8\delta C$ (if $m < 4\delta C$, we can store the entire graph in $\tilde{O}(C)$ bits of space). Moreover, Z_i for any given i is fairly concentrated around its expectation, since Theorem 2.7 implies that h is also $\Theta(C\delta)$ -wise independent, so we can bound its deviation using the generalized version of the Chernoff bound given by Lemma 2.8. Consider any Z_i for which $4\delta C \leq \mathrm{E}[Z_i] \leq 8\delta C$. Then we have

$$\Pr(Z_i < 3\delta C \text{ or } Z_i > 9\delta C) \le \Pr(|Z_i - \operatorname{E}[Z_i]| > \delta C) \le \Pr(|Z_i - \operatorname{E}[Z_i]| > \frac{1}{8}\operatorname{E}[x])$$

$$\le \exp(-\left\lfloor (\frac{1}{8})^2 \frac{\operatorname{E}[Z_i]}{3} \right\rfloor) \le \exp(-\Theta(C))$$

Using that $C \geq n$, the error probability is $\exp(-n)$. We also run an ℓ_0 -estimation algorithm, to estimate each Z_i up to a multiplicative factor of 1.1, with total space $\tilde{O}(1)$ and update time $\tilde{O}(1)$ [5], and with failure probability $1/n^{\Omega(1)}$. It follows from the above that for an i for which $4\delta C \leq \mathrm{E}[Z_i] \leq 8\delta C$, we have that $Z_i \in [3\delta C, 9\delta C]$ with probability $1 - \exp(-n)$. It follows that our ℓ_0 -estimate for this value of Z_i will be in $[C, 10\delta C]$ with probability $1 - 1/n^{\Omega(1)}$. Hence, from sp_i , with probability $1 - 1/n^{\Omega(1)}$ we will recover all values that hash to sp_i , and as argued above these are within statistical distance $e^{-\delta}$ from uniform.

Combining Lemma 2.9 with Theorem 2.5 proves Theorem 1.1.

To the best of our knowledge, the fastest known 0.5-approximation algorithm for the densest subgraph problem has a running time of O(m+n). However, here we need to find the densest subgraph of a sampled graph with at most $C = \tilde{O}(n)$ edges. Thus, the running time of our algorithm is $\tilde{O}(n)$. To the best of our knowledge, this is the fastest $(0.5 - \epsilon)$ -approximation algorithm for the densest subgraph problem. Theorem 1.2 states this fact.

3 A General Family of Problems

Here, we extend our results to the heavy subgraph problems. Specifically, we show that given an offline α -approximation algorithm for a (γ, l) -heavy subgraph problem, $\mathcal{P}(\gamma, l)$, Algorithm 2 is an $(\alpha - \epsilon)$ -approximation for $\mathcal{P}(\gamma, l)$. Later, in Section 4, we show several applications of this algorithm. In this section, we denote the solution in Sol_G^k that maximizes f by opt(G, k).

Algorithm 2 A General Algorithm

Input: A graph G, a heavy subgraph problem $\mathcal{P}(\gamma, l)$ and an α approximation algorithm Alg for \mathcal{P}

Output: An $\alpha - \epsilon$ estimator of \mathcal{P} on graph G, w.pr. $1 - m^{-\delta}$.

- 1: Set $C = \frac{12n(4+\delta)\log(l)}{\gamma\epsilon^2}$
- 2: if $|E| \leq C$ then
- 3: Return Alg(G).
- 4: else
- 5: Sample C edges uniformly at random, without replacement from G.
- 6: Let H be the sampled graph.
- 7: Return $\frac{1}{p}Alg(H)$.

The following lemma is the generalized version of Lemma 2.3.

Lemma 3.1. Let G be the input graph and let $\mathcal{P}(\gamma, l)$ be a heavy subgraph problem and let Alg be an α -approximation algorithm for problem \mathcal{P} . Let $H = (V, E_H)$ be the sampled graph by Algorithm 2 and let $p = \frac{C}{m}$. We have

$$\Pr\left(\alpha opt(G,k) - f_G(Alg(H))\right) \ge \epsilon opt(G)) \le 6 \exp(\log(|Sol_G^k|) - \frac{p\epsilon^2 opt(G)}{12f_k}),$$

where $f_G(Alg(H))$ is the objective value of G on a solution sol_G such that $Alg(H) = sol_G \cap H$.

Proof. We prove this lemma in a similar way to that of Lemma 2.3. Again here we define x_e to be the random variable that indicates whether e exists in E_H or not. However, here we let sol_G be an arbitrary solution from Sol_G^k . Let sol_H be a spanning subgraph of sol_G that contains the edges that appear in both sol_G and H. The hereditary property says that $sol_H \in Sol_H^k$.

By definition, the number of edges in sol_H is $\sum_{e \in sol_H} 1 = \sum_{e \in sol_G} x_e$. We denote this summation by X. Using the local linearity property we have

$$E[f(sol_H)] = E[f_k \sum_{e \in sol_G} x_e] = f_k \sum_{e \in sol_G} E[x_e] = f_k \sum_{e \in Sol_G} p = p \cdot f_k \sum_{e \in Sol_G} 1 = p \cdot f(sol_G).$$
 (5)

where the first and the last equalities are by Local Linearity, and the third equality is by definition of p. Again we can use Lemma 2.2 to claim that the random variables x_1, x_2, \ldots, x_m are negatively correlated. Thus, to bound X_U , we can apply the following form of the Chernoff bound from Lemma 2.1.

$$\Pr\left(|X - \mathrm{E}[X]| \ge \epsilon' \, \mathrm{E}[X]\right) \le 3 \exp(-\epsilon'^2 \, \mathrm{E}[X]/3)$$

By setting $\epsilon' = p\epsilon \frac{opt(G)}{2f_k \to [X]}$ we have

$$\begin{split} \Pr\left(|X - \mathbf{E}[X]| &\geq \frac{p\epsilon opt(G)}{2f_k}\right) \leq 3\exp(-\frac{p^2\epsilon^2 opt(G)^2}{4f_k^2 \, \mathbf{E}[X]^2} \cdot \frac{\mathbf{E}[X]}{3}) \\ &\leq 3\exp(-\frac{p^2\epsilon^2 opt(G)^2}{12f_k^2 \, \mathbf{E}[X]}) \\ &\leq 3\exp(-\frac{p\epsilon^2 opt(G)}{12f_k}) \end{split} \qquad \text{Using Equality 5} \end{split}$$

On the other hand we have

$$\Pr\left(|X - \mathbf{E}[X]| \ge \frac{p\epsilon opt(G)}{2f_k}\right) = \Pr\left(\frac{1}{p}f_k|X - \mathbf{E}[X]| \ge \frac{\epsilon}{2}opt(G)\right)$$

$$= \Pr\left(|\frac{1}{p}f_kX - \frac{1}{p}f_k\mathbf{E}[X]| \ge \frac{\epsilon}{2}opt(G)\right)$$

$$= \Pr\left(|\frac{1}{p}f(sol_H) - f(sol_G)| \ge \frac{\epsilon}{2}opt(G)\right) \quad \text{Using Equality 5}$$

Therefore, we have

$$\Pr\left(\left|\frac{1}{p}f(sol_H) - f(sol_G)\right| \ge \frac{\epsilon}{2}opt(G)\right) \le 3\exp\left(-\frac{p\epsilon^2opt(G)}{12f_k}\right) \tag{6}$$

If we set sol_G to be the vertex set of Opt(G,k), Inequality 6 states that

$$\Pr\left(opt(G,k) - \frac{1}{p}f_H(opt(G,k)) \ge \frac{\epsilon}{2}opt(G)\right) \le 3\exp(-\frac{p\epsilon^2opt(G)}{12f_k})$$

which immediately gives us

$$\Pr\left(opt(G,k) - \frac{1}{p}opt(H,k) \ge \frac{\epsilon}{2}opt(G)\right) \le 3\exp\left(-\frac{p\epsilon^2 opt(G)}{12f_k}\right). \tag{7}$$

On the other hand, Inequality 6 states that for each selection of sol_G , with probability $1-3\exp(-\frac{p\epsilon^2 opt(G)}{12f_k})$, we can upper bound $\frac{1}{p}f(sol_H)$ by $f(sol_G)+\frac{\epsilon}{2}opt(G)$. Indeed, we have $|Sol_G^k|$ such choices. Thus, by applying a union bound we have

$$\Pr\left(\forall_{sol_G \in Sol_G^k} \frac{1}{p} f(sol_H) - f(sol_G) \ge \frac{\epsilon}{2} opt(G)\right) \le 3|Sol_G^k| \exp(-\frac{p\epsilon^2 opt(G)}{12f_k}).$$

If we select sol_G , using the hereditary property, such that $sol_H = Alg(H)$, we have

$$\Pr\left(\frac{1}{p}Alg(H) - f_G(Alg(H)) \ge \frac{\epsilon}{2}opt(G)\right) \le 3|Sol_G^k|\exp(-\frac{p\epsilon^2opt(G)}{12f_k}).$$

Now, given that $Alg(H) \ge \alpha Opt(H)$, we have

$$\Pr\left(\frac{1}{p}opt(H) - \frac{1}{\alpha}f_G(Alg(H)) \ge \frac{1}{\alpha}\frac{\epsilon}{2}opt(G)\right) \le 3|Sol_G^k|\exp(-\frac{p\epsilon^2opt(G)}{12f_k}). \tag{8}$$

Therefore, by combining Inequalities 7 and 8 and applying the union bound we have

$$\Pr\left(opt(G,k) - \frac{1}{\alpha}f_G(Alg(H))\right) \ge \frac{1}{\alpha}\epsilon opt(G)\right) \le 3(|Sol_G^k| + 1)\exp(-\frac{p\epsilon^2 opt(G)}{12f_k})$$

$$\le 6\exp(\log(|Sol_G^k|) - \frac{p\epsilon^2 opt(G)}{12f_k})$$

Now, we are ready to prove Theorem 1.3.

proof of Theorem 1.3. Lemma 3.1 together with Equality 5 imply that for each k, with probability at least $1-6\exp(\log(|Sol_G^k|)-\frac{p\epsilon^2opt(G)}{12f_k})$ we have $\alpha opt(G,k)-\frac{1}{p}Alg(H))\geq \epsilon opt(G)$. By a union bound, this holds for all $1\leq k\leq l$, with probability $1-\sum_{k=1}^l 6\exp(\log(|Sol_G^k|)-\frac{p\epsilon^2opt(G)}{12f_k})$. Thus, for some k with probability $1-6l\cdot\exp(\log(|Sol_G^k|)-\frac{p\epsilon^2opt(G)}{12f_k})$ we have $\alpha opt(G)-\frac{1}{p}Alg(H))\geq \epsilon opt(G)$, which means that Algorithm 2 outputs a $(1-\epsilon)$ -approximation.

Now, if we set p to $\frac{12n(4+\delta)\log(l)}{\gamma\epsilon^2m}$, or equivalently set C in Algorithm 2 to $\frac{12n(4+\delta)\log(l)}{\gamma\epsilon^2}$, we have

$$\begin{split} &1 - 6l \cdot \exp(\log(|Sol_G^k|) - \frac{p\epsilon^2 opt(G)}{12f_k}) \\ &= 1 - 6l \cdot \exp(\log(|Sol_G^k|) - \frac{12n(4+\delta)\log(l)}{\gamma\epsilon^2 m} \frac{\epsilon^2 opt(G)}{12f_k}) \\ &\leq 1 - 6l \cdot \exp(\log(|Sol_G^k|) - \frac{12n(4+\delta)\log(l)}{\gamma\epsilon^2 m} \frac{\epsilon^2}{12f_k} \gamma \log(|Sol_G^k|) f_k \frac{m}{n}) \quad \text{From } \gamma \text{ Bound} \\ &= 1 - 6l \cdot \exp(\log(|Sol_G^k|) - (4+\delta)\log(l)\log(|Sol_G^k|)) \\ &< 1 - \exp(\log(l) + 2 + \log(|Sol_G^k|) - (4+\delta)\log(l)\log(|Sol_G^k|)) \\ &\leq 1 - \exp(-\delta\log(l)\log(|Sol_G^k|)) \\ &= 1 - e^{-\delta} \end{split}$$

4 Applications

There are several problems that fit into the class of heavy subgraph problems. Some examples are densest bipartite subgraph, directed densest subgraph, d-max cut, and d-sum-max clustering. In this section we define each of these problems and prove that each satisfies the properties required of a heavy subgraph problem.

4.1 Densest Bipartite Subgraph

In the densest bipartite subgraph problem, we are given a graph general G and we aim to find a bipartite subgraph sol of G with the maximum density. Let opt be a densest bipartite subgraph of G, with parts A_{opt} and B_{opt} . Then opt contains all edges of G that are between A_{opt} and B_{opt} . We call such a subgraph feasibly maximal and without loss of generality, restrict all of the solutions to be feasibly maximal. In fact, we can indicate a feasibly maximal solution sol by its two parts A_{sol} and B_{sol} .

proof of First part of Theorem 1.4. For normalization purposes, we increase the value of the objective function by a factor of n. Without loss of generality define the density to be $n \cdot \frac{|E_{sol}|}{|V_{sol}|}$. We set l=n, and for any $1 \leq k \leq l$, we let Sol_G^k be the set of all solutions sol such that $|A_{sol}| + |B_{sol}| = n - k + 1$. Thus, we have $|Sol_G^k| = \binom{n}{n-k+1} 2^{n-k+1}$.

Local Linearity: The density of a solution sol in Sol_G^k is $n\frac{|E_{sol}|}{n-k+1} = |E_{sol}|\frac{n}{n-k+1}$. Thus, we can set $f_k = \frac{n}{n-k+1}$, which is increasing in k and we have $f_1 = \frac{n}{n-1+1} = 1$, as desired.

Hereditary Property: Let H be a spanning subgraph of G. For any solution $sol \in Sol_G^k$, the intersection of sol and H remains bipartite and feasibly maximal, and thus, is a solution in Sol_H . Moreover, by definition, the number of vertices of this intersection is the same as sol. Thus, it belongs to Sol_H^k . On the other hand, for any solution $sol \in Sol_H^k$ with parts A_{sol} and B_{sol} , let sol' be the bipartite maximal subgraph of G between the partitions A_{sol} and B_{sol} . By definition, $sol' \in Sol_G^k$ and clearly sol' satisfies $sol = sol' \cap H$.

 γ Bound: In fact, G contains a bipartite subgraph that contains at least $\frac{m}{2}$ edges. The density of this subgraph is at least $n\frac{m/2}{n}=\frac{m}{2}$. On the other hand we have, $f_k=\frac{n}{n-k+1}$ and $|Sol_G^k|=\binom{n}{n-k+1}2^{n-k+1}$. Thus, if we set γ to $\frac{2}{\log(n)+1}$, we have

$$\gamma \log(|Sol_G^k|) f_k \frac{m}{n} = \frac{2}{\log(n) + 1} \log(\binom{n}{n - k + 1} 2^{n - k + 1}) \frac{n}{n - k + 1} \frac{m}{n}$$

$$\leq \frac{2}{\log(n) + 1} ((n - k + 1) \log(n) + (n - k + 1)) \frac{n}{n - k + 1} \frac{m}{n}$$

$$\leq \frac{2}{\log(n) + 1} (\log(n) + 1) m$$

$$= \frac{m}{2} \leq opt.$$

4.2 Directed Densest Subgraph

In the directed densest subgraph problem we are given a directed graph G, we want to find two not necessarily disjoint sets $A, B \subseteq V_G$, to maximize $\frac{|E(A,B)|}{\sqrt{|A|\cdot|B|}}$, where, E(A,B) is the set of all edges $(u,v) \in E_G$, such that $u \in A$ and $v \in B$.

proof of Second part of Theorem 1.4. For normalization, we increase the objective function by a factor of n and define it as $n \frac{E(A,B)}{\sqrt{|A|\cdot|B|}}$. For simplicity, here we index the solution sets using a pair of indices i and j. Here, $Sol_G^{i,j}$ contains any solution sol = (A,B) such that i = |A| and j = |B|. Thus, we have $|Sol_G^{i,j}| = \binom{n}{i}\binom{n}{i}$, and $l = n^2$.

Local Linearity: In fact, for a solution $sol = (A, B) \in Sol_G^{i,j}$, the directed density of sol is $n\frac{|E(A,B)|}{\sqrt{|A|\cdot |B|}} = n\frac{|E(A,B)|}{\sqrt{i\cdot j}} = |E(A,B)|\frac{n}{\sqrt{i\cdot j}}$. Thus, we can define, $f_{i,j} = \frac{n}{\sqrt{i\cdot j}}$, and we have $min_{i,j}(f_{i,j}) = \frac{n}{\sqrt{n\cdot n}} = 1$, as desired.

Hereditary Property: For any spanning subgraph $H \subseteq G$, and any solution $sol = (A, B) \in Sol_G^{|A|,|B|}$, the same sets A and B indicate the intersection of H and sol, and thus is a solution in $Sol_H^{|A|,|B|}$. On the other hand, for any solution $sol \in Sol_H^{|A|,|B|}$ with sets A and B, let sol' be the solution on G corresponds to the sets A and B. By definition, $sol' \in Sol_G^{|A|,|B|}$ and clearly sol' satisfies $sol = sol' \cap H$.

 γ Bound: The directed density of the solution $sol = (V_G, V_G)$ is $n \frac{m}{\sqrt{n \cdot n}} = m$. Therefore, the optimum is lower bounded by m. On the other hand, for any i and j we have $|Sol_G^{i,j}| = \binom{n}{i} \binom{n}{j}$ and $f_{i,j} = \frac{n}{\sqrt{i \cdot j}}$. If we set γ to $\frac{1}{2\sqrt{n}\log(n)}$ we have

$$\gamma \log(|Sol_G^{i,j}|) f_{i,j} \frac{m}{n} = \gamma \log(\binom{n}{i} \binom{n}{j}) \frac{n}{\sqrt{i \cdot j}} \frac{m}{n}$$

$$\leq \gamma (i \cdot \log(n) + j \cdot \log(n)) \frac{n}{\sqrt{i \cdot j}} \frac{m}{n}$$

$$= \gamma \frac{i \cdot \log(n) + j \cdot \log(n)}{\sqrt{i \cdot j}} m$$

$$= \gamma (\frac{\sqrt{i} \cdot \log(n)}{\sqrt{j}} + \frac{\sqrt{j} \cdot \log(n)}{\sqrt{i}}) m$$

$$\leq \gamma 2 \sqrt{n} \log(n) m$$

$$= \frac{1}{2\sqrt{n} \log(n)} 2 \sqrt{n} \log(n) m$$

$$= m \leq opt.$$

4.3 d-Max Cut

In the d-max cut problem, we are given a graph G and are supposed to mark the vertices using d labels to maximize the number of edges with different labels.

proof of Third part of Theorem 1.4. Here we simply let all the solutions be in Sol_G^1 . Indeed, we have l=1 and $|Sol_G^1|=d^n$.

Local Linearity: Clearly we have $f_1 = f_l = 1$.

Hereditary Property: For any spanning subgraph $H \subseteq G$, and any solution $sol \in Sol_G^1$, the same labeling of sol on H gives us the intersection of sol and H. Thus, the intersection of sol and H is a solution in $Sol_H = Sol_H^1$. On the other hand, similarly, for any solution $sol \in Sol_H^1$, the same labeling gives us a solution $sol' \in Sol_G^1$ such that $sol = sol' \cap H$.

 γ Bound: Again here, if we just use 2 labels we have a solution with $\frac{m}{2}$ value. Thus, we have $opt \geq \frac{m}{2}$. If we set γ to $\frac{1}{2\log(d)}$ we have

$$\gamma \log(|Sol_G^k|) f_k \frac{m}{n} = \frac{1}{2 \log(d)} \log(d^n) \frac{m}{n} = \frac{1}{2 \log(d)} \log(d) m = \frac{m}{2} \le opt.$$

4.4 d-Sum-Max Clustering

This problem is fairly similar to d-max cut. Again we are given a graph G and are supposed to mark the vertices using d labels. However, here we have to use all d colors and want to maximize the number of edges with the same labels.

proof of Fourth part of Theorem 1.4. Again, here we simply let all the solutions be in Sol_G^1 . So we have l=1 and $|Sol_G^1| \leq d^n$.

Local Linearity: We have $f_1 = 1$.

Hereditary Property: Consider a spanning subgraph $H \subseteq G$, and let $sol \in Sol_G^1$ be an arbitrary solution. Again, we can use the same labeling as sol on H to get the intersection of sol and H. Thus, the intersection of sol and H is a solution in Sol_H^1 . On the other hand, again, for any solution $sol \in Sol_H^1$, the same labeling gives us a solution $sol' \in Sol_G^1$ such that $sol = sol' \cap H$.

 γ Bound: Suppose we choose d-1 vertices uniformly at random and label them with labels $1,2,\ldots,d-1$ and label all the other vertices with d. Then the probability that one of the endpoints of a fixed edge is not labeled by d is at most $2\frac{d-1}{n}$. Thus, the expected number of edges in such a solution is $m-2m\frac{d-1}{n}=m\frac{n-2(d-1)}{n}\geq m\frac{n-2d}{n}$. Thus, the optimum solution has at least $m\frac{n-2d}{n}$ edges. Now, if we set γ to $\frac{n-2d}{n\log(d)}$ we have

$$\gamma \log(|Sol_G^k|) f_k \frac{m}{n} \le \frac{n-2d}{n \log(d)} \log(d^n) \frac{m}{n} = \frac{n-2d}{n \log(d)} \log(d) m = m \frac{n-2d}{n} \le opt.$$

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